METASTABLE STATES IN TRANSPORT PROCESSES DESCRIBED BY A QUASILINEAR PARABOLIC EQUATION

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1. There is considerable discussion at the present time in regard to the solutions of a quasilinear parabolic equation of the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial u^k}{\partial x}\right|^{n-1}\frac{\partial u^k}{\partial x}\right)-\gamma u^m\equiv Lu,\ k,\gamma>0,\ m\geqslant 0,\ kn>1, \tag{1.1}$$

which is used to describe various nonlinear transport processes. The dependence of the transport and absorption coefficients on the transported quantity u and its gradient $\partial u/\partial x$ is approximated here by a power function. In particular, for $n=1$ and $k\neq 1$ Eq. (1.1) coincides with the nonlinear heat-conduction equation discussed in [1]; for $k=1$ and $n\neq 1$ it is the momentum transport equation for a nonlinearly viscous non-Newtonian fluid [2]; the case $k=2$, $m=0$ corresponds to the motion of the indicated electrically conducting fluid in a laminar boundary layer in a transverse magnetic field [3]. In the general case of arbitrary n, k, and m Eq. (1.1) is known as the turbulent filtration equation [1, 4] with a nonlinear sink.

An important feature of the transport processes described by Eq. (1.1) is the possible existence of a frontal surface $S(x, t)=0$ separating the region with $u(x, t)=0$ and the region of spatial localization of the transported quantity, where $u(x, t)>0$ (see, e.g., [1-4]). The form of the function $S=S(x, t)$ in the case of the Cauchy problem for Eq. (1.1) has been investigated in [5], where, following [6, 7], the authors demonstrate the possibility of metastable states of the solution. In such a state, during a finite time interval $t\in[0, T]$ the function $S(x, t)$ depends only on the coordinate x, $S(x, t)\equiv S(x)$.

Metastable states are possible, in particular, in the transition from one steady-state solution to another steady-state solution. In this connection we demonstrate the possible existence of metastable states in a boundary-value problem for Eq. (1.1) and give bounds on the duration of a metastable state, which are confirmed by direct numerical calculations.

We consider Eq. (1.1) on the set

$$G=R_-\times R_+=\{(x,\ t): x\in R_-,\ t\in R_+\}, \tag{1.2}$$

where $R_+=\{t: t\geqslant 0\}$, $R_-=\{x: x\leqslant 0\}$.

We denote by $\Omega=\{(x,\ t)\in G;\ u(x,\ t)>0\}$ the region of localization of the transported quantity, $G\setminus\Omega=\{(x,\ t)\in G:\ u(x,\ t)=0\}$. We specify that the boundary condition is monotonic and bounded:

$$u(0,\ t)=\varphi(t),\ \varphi(t_2)\geqslant\varphi(t_1),\ \text{if}\ t_2>t_1,\ U_0=\max\varphi(t)<\infty. \tag{1.3}$$

We assume that the initial condition is finite and specify it in the "natural" form

$$u(x,0)\equiv u_0(x)=\begin{cases}A\left(1-\dfrac{x}{x_0}\right)^{\frac{n+1}{kn-m}}, & x_0<x\leqslant 0,\ kn>m,\\ 0, & -\infty<x\leqslant x_0,\end{cases} \tag{1.4}$$

where

$$A=\left[\frac{\gamma(kn-m)^{n+1}(-x_0)^{n+1}}{n(k+m)(kn+k)^n}\right]^{1/(kn-m)},$$

and the function $u(x, 0)$ satisfies the differential equation $Lu_0=0$. We assume that in the region Ω the derivative $\partial u^k/\partial x\geq 0$; under this condition Eq. (1.1) is written in the form

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\frac{\partial u^k}{\partial x}\right)^n+\gamma u^m=0. \tag{1.5}$$

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We show that the solution of problem (1.3)-(1.5) is metastably localized and the metastable localization time T satisfies the inequality

$$T \geqslant \frac{U_0^{kn-m}}{A_1^{kn-1}} \frac{(k+m)(kn-1)^n}{\gamma(k+1)(kn-m)^{n+1}}, \text{ если } m \in [1,\ kn[,\ A_1(k,\ n,\ t_0,\ x_0) = \text{const}.$$

We first prove that the solution of problem (1.3)-(1.5) is metastably localized if $m \in [1, kn]$. We investigate the auxiliary problem for an equation of the form

$$\frac{\partial \omega}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\partial \omega^k}{\partial x}\right)^n = 0 \tag{1.6}$$

with the initial condition

$$\omega(x,\ 0) = A_1(1 - x/x_0)^{(n+1)/(kn-1)},\quad x \in]x_0,\ 0] \tag{1.7}$$

and the boundary condition

$$\omega(0,\ t) = A_1(1 - t/t_0)^{-1/(kn-1)}, \tag{1.8}$$

$$A_1 = \left\{\left[\frac{kn-1}{kn+k}\right]^n \frac{(-x_0)^{n+1}}{n(k+1)t_0}\right\}^{\frac{1}{kn-1}},\ \omega(x_0,\ t) = 0,\left(\frac{\partial \omega^k}{\partial x}\right)^n(x_0,\ t) = 0.$$

The solution of the auxiliary problem (1.6)-(1.8) has the form [6]

$$\omega(x,\ t) = \begin{cases} A_1\left[\left(1 - \frac{x}{x_0}\right)^{n+1}\left(1 - \frac{t}{t_0}\right)^{-1}\right]^{\frac{1}{kn-1}}, \\ 0, \end{cases} \tag{1.9}$$

where $t \in [0, t_0]$, $x \in [x_0, 0]$. It is seen that the solution (1.9) is metastably localized for $t \in [0, t_0[$. Because of the monotonic behavior of the solution $u(x, t)$ of problem (1.3)-(1.5) as a function of the initial and boundary condition and the fact that $\gamma > 0$ the function (1.9) majorizes the solution of this problem with the appropriate specification of the parameter t_0, inferred from a comparison of $U_0 = A$ and A_1. Consequently, the solution of problem (1.3)-(1.5) is metastably localized if $m \in [1, kn[$.

From a comparison of the boundary conditions (1.3) and (1.8) we can deduce a lower bound on the metastable localization time in problem (1.3)-(1.5) $t \in [0, T]$, $T \geq t_0$:

$$T \geqslant t_0 = \left(\frac{U_0^{kn-m}}{A_1^{kn-1}}\right)\frac{(k+m)(kn-1)^n}{\gamma(kn-m)^{n+1}(k+1)}. \tag{1.10}$$

It can be shown with the aid of the comparison theorem [8] that the boundary of the region of localization in problem (1.3)-(1.5) is necessarily set in motion, $S(x, t) \neq S(x)$, for $t > t_0$, i.e., the solution of the problem is indeed metastable. This is most simply accomplished in the case $\gamma = 0$. Accordingly, we consider the boundary problem for Eq. (1.6) for the function $\omega_1 = \omega_1(x, t)$ with the boundary and initial conditions

$$\omega_1(x,\ 0) = 0,\ \omega_1(0,\ t) = U_1 = \text{const} > 0,\ U_1 < \min \varphi(t). \tag{1.11}$$

The boundary-value problem (1.6) (1.11) for $\gamma = 0$ is self-similar [9]. If we introduce the self-similar variable $\eta = x/x_f(t)$, where $x_f(t)$ is the boundary of the support of the solution and is given by the equation $S(x_f(t), t) = 0$, along with the new dependent variable $\omega_1 = U_1 f_1(\eta)$, we reduce problem (1.5), (1.11) to

$$\beta\eta \frac{df_1}{d\eta} + \frac{d}{d\eta}\left[\operatorname{sign} x_f \frac{df_1^k}{d\eta}\right]^n = 0,\ \eta \in [0,\ 1],\ f_1(0) = 1,\ f_1(1) = 0. \tag{1.12}$$

From the self-similarity condition we have

$$x_f = -\left[(n+1)\beta U_1^{kn-1} t\right]^{1/(n+1)}.$$

We evaluate the constant $\beta > 0$ in problem (1.12) from the condition of zero net flow at the front $df_1^k(1)/d\eta = 0$. The results of a numerical calculation of the parameter $\beta = \beta(k, n)$ in the special case $n = 1$, obtained by the collocation method after preliminary quasilinearization, are shown in Fig. 1.

By virtue of the comparison theorem [8] we have the inequalities $u(x, t) > \omega_1(x, t)$, $U_1 < \min \varphi(t)$. Consequently, from the inequality $x_f(t_1) \leq x_0$ we deduce an upper bound for the metastable localization time of the solution: $T \leqslant t_1 = (-x_0)^{n+1}/\beta(n+1)U_1^{kn-1}$.

The existence of metastable states for the function $\omega_2 = \omega_2(x, t)$ in the case $\gamma \neq 0$ can be demonstrated by solving numerically the boundary-value problem comprising Eq. (1.6), the conditions

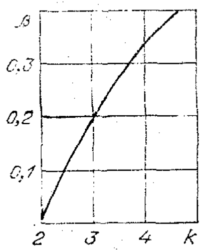

Fig. 1

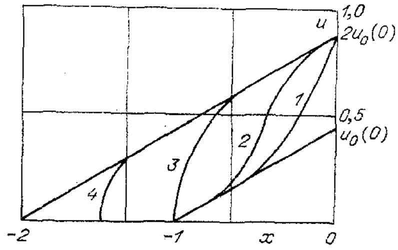

Fig. 2

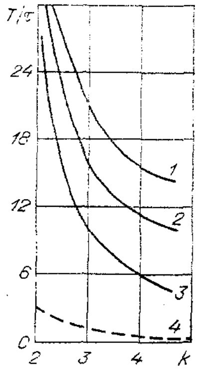

Fig. 3

$$\omega_2(0,\ t) = \psi(t),\ \psi(t_2) \geqslant \psi(t_1),\quad \text{if}\quad t_2 > t_1,\ \varphi(t) > \psi(t), \tag{1.13}$$

and the initial condition (1.4). On the basis of the comparison theorem, with respect to the initial and boundary conditions the solution of problem (1.3)-(1.5) majorizes the solution of problem (1.4), (1.5), (1.13), $u(x, t) \geq \omega_2(x, t)$. If $k > 1 + 1/n$, the numerical solution of problem (1.4), (1.5), (1.13) shows that for times $t > T$ the first-order discontinuity front is set in motion. Consequently, the solution of problem (1.3)-(1.5) is metastable.

2. To confirm the foregoing bounds we have carried out some numerical calculations. We used an implicit differencing scheme [10]. The position of the front was determined approximately where the solution acquired the order of magnitude of the computational error. The time step τ and the space step h were made equal to 0.0364 and 0.04 respectively. As an example, Fig. 2 shows the evolution of the solution of problem (1.3)-(1.5) for $n=3$, $k=2$, $\gamma=1$, $m=2$, $x_0=-1$, $\varphi(t)=2u_0(0)=0.878$, with the curves numbered as follows: 1) $t=\tau$; 2) $t=3\tau$; 3) $t \equiv T = 10\tau$; 4) $t = 20\tau$. It is evident from the graph that the time of onset of motion of the surface can be determined with error O(h) from the variation of the derivative $\partial u/\partial x$ as $x \to x_f(t)_{+0}$. As $t \to \infty$ the solution goes over to a new steady state, corresponding to the altered boundary condition.

It is clear that the duration T of the metastable state depends considerably on the form of the function $\varphi(t)$, given identical values of $\varphi(0)$ and $\varphi(\infty)$. Accordingly, we have calculated the duration T of the metastable state for various functions $\varphi = \varphi(t)$. The results are shown in Fig. 3 for $n=1$, $\gamma=1$, $m=\lambda$, $x_0=-1$. The curves are numbered as follows:

$$\varphi(t) = \begin{cases} u_0(0)\ \left(1+\frac{t}{4}\right),\ t \in \Gamma, \\ 2u_0(0),\ t \overline{\in} \Gamma; \end{cases}$$

$$\varphi(t) = \begin{cases} u_0(0)\exp\left(\frac{\tau t}{4h}\right),\ t \in \Gamma, \\ 2u_0(0),\ t \overline{\in} \Gamma; \end{cases}$$

$$\varphi(t) = 2u_0(0),\ t \in \Gamma,$$

where $\Gamma = \{t : t > 0,\ \varphi(t) < 2u_0(0)\}$. The dashed curve in Fig. 2 corresponds to the lower bound of the duration of the metastable state according to relation (1.10). All the numerical calculations exhibit good consistency with the derived bounds and corroborate the qualitative analysis of the problem.

LITERATURE CITED

1. Ya. B. Zel'dovich and A. S. Kompaneets, "Theory of heat propagation in temperature-dependent heat conduction," in: In Honor of the Seventieth Birthday of A. F. Ioffe [in Russian], Izd. Akad. Nauk SSSR, Moscow (1950).
2. W. L. Wilkinson, Non-Newtonian Fluids: Fluid Mechanics, Mixing, and Heat Transfer, Pergamon, New York (1960).
3. K. B. Pavlov, "Magnetohydrodynamic flow of an incompressible viscous fluid due to deformation of a plane surface," Magn. Gidrodin., No. 4 (1974).
4. G. I. Barenblatt, "Some unsteady motions of a liquid and a gas in a porous medium," Prikl. Mat. Mekh., 16, No. 1 (1952).
5. K. B. Pavlov and A. S. Romanov, "Variation of the region of localization of disturbances in nonlinear transport processes," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1980).

6. A. A. Samarskii, N. B. Smitrenko, S. P. Kurdyumov, and A. P. Mikhailov, "The effect of metastable heat localization in a medium with a nonlinear thermal conductivity," Dokl. Akad. Nauk SSSR, 223, No. 6 (1975).
7. A. S. Kalashnikov, "Nature of the propagation of disturbances in nonlinear heat-conduction problems with absorption," Zh. Vychisl. Mat. Mat. Fiz., 14, No. 4 (1974).
8. L. D. Pokrovskii and S. N. Taranenko, "Spatial localization of the solutions of nonlinear equations of parabolic type," in: Collected Works of the Department of Higher Mathematics, Dedicated to the 150th Year of Moscow Technical University [in Russian], No. 336 (1980).
9. G. I. Barenblatt, "Self-similar motions of a compressible fluid in a porous medium," Prikl. Mat. Mekh., 16, No. 6 (1952).
10. A. A. Samarskii, Theory of Differencing Schemes [in Russian], Nauka, Moscow (1977).

STRUCTURES OF THE CONJUGATE SATURATION AND CONCENTRATION DISCONTINUITIES IN THE DISPLACEMENT OF OIL BY A SOLUTION OF AN ACTIVE MATERIAL

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A general description of the displacement of oil by a solution of an active material not only in the basic case of a single active factor, but also in more complicated situations is presented in [1-5]. Here a central part is played by the scope for constructing a solution in a large-scale approximation, i.e., neglecting diffusion processes of various types (capillarity, diffusion proper, and thermal conductivity). These processes have marked effects on the solution only in zones where the variables alter sharply, which correspond to discontinuities in the large-scale approximation. Here we examine the fine structure of the transition zones. The results may be of value in estimating the limits to the application of the large-scale approximation and to the failure times for the layer of active material, as well as in developing numerical and approximate methods.

1. Formulation: External Solution. We consider the one-dimensional frontal displacement of oil by a solution of an active material. We write the equations for the phase infiltration law ($i=1$ for water and $i=2$ for oil) and the conservation equations for water, oil, and the active material on the basis that the mass concentrations of the material in the water c and in the oil φ are small, while the porosity m, permeability k, and phase densities ρ_1 and ρ_2 are constant:

$$u_i = -(kf_i(s,\ c)/\mu_i(c))\partial p_i/\partial x \ (i = 1,\ 2),$$
$$p_2 - p_1 = p = \gamma(c)J(s),$$
$$m\partial s/\partial t + \partial u_1/\partial x = 0,\ -m\partial s/\partial t + \partial u_2/\partial x = 0,$$
$$m\frac{\partial}{\partial t}[\varkappa cs + \varphi(c)(1-s) + a(c)] + \frac{\partial}{\partial x}[\varkappa cu_1 + \varphi(c)u_2] = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right). \quad (1.1)$$

Here s is the water content; $m\rho_2 a$, mass of sorbed material in unit volume of the porous medium; f_i, μ_i, p_i, phase permeability, viscosity, and pressure for phase i; D, diffusion coefficient for the active material; p, capillary pressure, whose dependence on the surface tension incorporates the coefficient $\gamma(c)$; J, a Leverett function; x, coordinate; t, time; and $\varkappa = \rho_1/\rho_2$.

We introduce the dimensionless variable

$$x' = x/L,\ t' = u_0t/L,\ u_i' = u_i/u_0,\ p_i' = p_i/\Delta p,\ u_0 = k\Delta p/\mu_1(0)L,$$
$$\mu_i' = \mu_i/\mu_1(0),\ D' = D/D_0,\ \gamma'(c) = \gamma(c)/\gamma(0),\ \varepsilon = \gamma(0)/\Delta p,\ \nu = D_0/u_0L,$$

where L, Δp, D_0 are the characteristic values of the size of the flow region, the external pressure difference, and the diffusion coefficient.

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